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Philosophical Origins in Mathematics?
Árpád Szabó Revisited

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I. The Internalist Historiographic Tradition

Since the turn of the 19th and 20th centuries when systematic research on antique Greek mathematics began, the general tone of historiographical work in the field has been dominated by the so-called internalist attitude. H.G. Zeuthen, P. Tannery and others initiated a research tradition whose goal is to reconstruct the mathematical development of the minus 5th and 4th century Greece by providing a technical and conceptual analysis of the available source texts, notably of Euclid's *Elements*. Such a reconstruction relies on today's conception of mathematical knowledge, inasmuch as it proceeds by decontextualising antique texts and separating their 'mathematical content' from the 'form of expression', the latter regarded as incidental and irrelevant. According to this view, the history of mathematics appears as a development of interrelated mathematical concepts and propositions, and the growth of mathematical knowledge lies in our increasing understanding of the network of these conceptual relations. Thus our better understanding of mathematics can, and should, shed light on the actual content of ancient mathematical texts by disregarding the 'clumsy' and 'cumbersome' language of expression and reformulating the original problems in the more suitable mathematical language of our modern age.

Notwithstanding some seriously problematic characteristics of this attitude to which I shall partly return, the internalist historiographic tradition developed an insightful and coherent interpretation of early Greek mathematics such as had never been available before. One of its key concepts is 'geometric algebra', which expresses the view that many of the mathematical problems solved in Euclid are genuinely algebraic problems dressed up in a 'geometric garb'. There are two reasons why the Greeks used a geometrical language to solve algebraic problems: (i) the supposedly scandalous discovery that some 'magnitudes' (quantities) cannot be measured with the same unit, i.e. they are 'incommensurable' and cannot be written as the ratio of any two integers, while there is no problem in representing them as line sections (e.g. the side and the diameter of a square); and (ii) the integration of the inherited Babylonian algebra-like mathematical knowledge into the earliest, mainly geometrical, tradition of Greek mathematics. Following this conception, a large portion of Euclid's (as well as Apollonius' and Archimedes') works can be re-interpreted as geometric algebra, or as algebraic problem-solving in a misleading and dispensable geometrical context. The emergence of the deductive, later axiomatic, mathematical style came as a result of a systematic reaction to the problem of incommensurability.

Before contrasting the internalist picture with Szabó's claims, let me highlight the platonist bias underlying its position. The task of relieving the essential mathematical content from the incidental form of expression in Greek texts presupposes the eternal existence or ahistorical validity of mathematical truths and entities, about which our mathematical knowledge is attained. In other words, ancient Greek mathematics and modern mathematics are believed to express the same truths, the same states of affairs belonging to some independent, of us and of our cognitive enterprises, Realm of Mathematics. According to this view, the history of mathematics must be seen as the process of discovering this mathematical realm, with ever better and more appropriate tools. To give a somewhat exaggerated illustration of this view, let me quote André Weil, mouthpiece of the essentially platonist Bourbaki-circle: "it is impossible for us to analyse properly the contents of Books V and VII of Euclid without the concept of group and even that of groups of operators, since the ratios of magnitudes are treated as a multiplicative group operating on the additive group of the magnitudes themselves".[1] To wit, those Greeks like Euclid with no concept of groups of operators in hand, were sleepwalkers who half-consciously formulated apt claims about the realm of truths long before the possibility for a precise description would appear.

II. Szabó's Views on Greek Mathematics

In the second half of the 20th century, some scholars like Sabetai Unguru emphasised the need for a different 'road' to approach Greek mathematics, one which puts the stress on the intrinsic coherency of texts on the one hand and, on the other hand, re-embeds mathematical knowledge in a wider cultural context within which it was created.[2] For this, the historian is invited to put aside modern mathematical knowledge as a context of interpretation and, rather, to exploit more tools of the historical kind for studying the development of mathematical views, together with other intellectual and cultural fields connected to mathematics in ancient Greece. In line with this attitude, Árpád Szabó performed a terminological and conceptual analysis of mathematical texts – an irrelevant and vain enterprise for those who dismiss the language of expression as unimportant and historically incidental. I summarise his results in the followings.[3]

In Szabó's view, the most striking characteristics of Greek mathematics is the deductive style of argumentation. This novelty, of which we see no traces in any previous culture interested in mathematics, is not just one of the many results of Greek mathematical activities. It is more than that: since the Greeks, deductive method has been, and probably will ever be, one the most fundamental features (if not 'the' most fundamental one) of mathematical thought. However, mathematics is not the only field where the importance of argumentative inquiry was recognised at that time. As we know it, the history of conscious argumentation started with the Eleatic philosophical school in the 5th century BC where the power of anti-empirical and purely intellectual arguments was first recognised. At least in the history of philosophy, clear evidences indicate that the Eleatic arguments shook the whole intellectual life of ancient Greece and challenged virtually every philosopher around. Szabó suggests that the origins of the deductive mathematical tradition reach back to the very same challenge, namely, it was the Eleatic dialectical school which fertilised the emerging mathematical knowledge and set up the anti-empirical framework within which classical Greek mathematics unfolded.

One of the possible evidences supporting this view is the excessive role of indirect proofs in Greek mathematics. It was the Eleatics who first constructed indirect arguments, and soon these became an essential tool in the field of dialectics. Now, it has been pointed out that there are more indirect proofs in Euclid than necessary, which shows some kind of 'enthusiasm' in using this form of argument. Moreover, the 'scandal' of incommensurability could not have appeared without indirect proofs, since showing that two magnitudes have no common measure requires indirect argumentation – which indicates that the development of deductive mathematical thought must have started even before the discovery of incommensurability, contrary to the widely accepted view. It is worth noting that later, at the highest stage of Greek mathematics around the time of Apollonius and Archimedes, indirect proofs acted as the life and soul of infinitesimal-like calculations. And this is just one example of the several probable connections between the Eleatics and mathematics: Szabó claims to find a number of conceptual relations between actual Eleatic views and the Pythagorean theory of numbers, for example. Yet his most plausible claims are those concerning the close terminological connections between the vocabulary of dialectics and the terms related to the axiomatic-deductive arsenal of mathematics (such as "definition", "axiom", "postulate", etc.).

In summary, Szabó suggests that deductive mathematics, and hence 'pure' mathematics as we call it (back to the point later), finds its origins in the intellectual revolution initiated by the Eleatic challenge in ancient Greece. To put it sharply, he claims that "systematic and deductive mathematics was, at the earliest stages of its history, a special branch of philosophy or, more precisely, of Eleatic dialectics."[4] Then he goes on to show how the Pythagorean theory of numbers developed from the Eleatic philosophy, partly in line with it and partly along essential critical alterations, until application of the deductive framework to geometry finally led Greeks to recognise the genuine features of mathematics that dissolve it from philosophy. Mathematics became an independent discipline around the turn of the 5th and 4th centuries BC, as a result of a relatively long independence war fought for its own foundations.

III. The Reconstructionist Criticism

While many historians agreed to Szabó's views and found them insightful, others within the internalist tradition strongly rejected them. Here I discuss only one type of criticism, but one which I find especially helpful in showing a number of misunderstandings about Szabó's conception.

As an illustration, let me first quote what W. Kneale, the famous historian of logic objected to the thesis concerning the connection between Eleatic logic and Greek mathematics: "I do not think we should assume that mathematicians cannot use a logically valid pattern of reasoning in their work until some philosopher has written about it and told them that it is valid. In fact we know that this is not the way in which the two studies, logic and mathematics, are related."[5] Indeed, the relation between logic and mathematics is conceived in a different form in our modern age. According to this view, mathematical knowledge is properly expressed only in formal axiomatic-deductive systems, and the general features of these systems (such as syntactical structure, rules of inference, etc.) are determined by logic. Actually, the role of logic in mathematics is to prescribe the correct framework within which mathematical theories are constructed, thus providing a meta-theory of mathematical theories.

This view of mathematics can be titled with the label of 'formalism', although here I use the term in a somewhat broader sense than it is usual in philosophy of mathematics. While platonism and formalism are two distinct, basically irreconcilable views for philosophers of mathematics, in the working mathematician's views they are often simultaneously present to some degree – as R. Hersh put it, "the typical mathematician is a platonist on weekdays and a formalist on Sundays".[6] No wonder, since platonism renders a promising explanation for the problem of content, i.e. to the desired objective validity of knowledge, while formalism explains the problem of form, i.e. the integrity of knowledge. Thus another bias underlying the internalist, 'mathematician's' approach to the history of mathematics, beside the platonist presupposition, is the formalist view of mathematics.

History of mathematics provides a peculiar field for combining the platonist and formalist presuppositions. No doubt that while logical metatheory should precede mathematical theory in the logical sense, the historical order is always reverse. This point is the principal motivation for K. Berka's criticism of Szabó: "It cannot be doubted that deductive systematization and conscious metatheoretical consideration presuppose a great amount of theoretical knowledge. That theory precedes metatheory is a fact well-known from the history of science…"[7] It is obvious that the logical 'knowledge' of the Eleatics was largely insufficient to serve as a metatheory of Greek mathematics. Not even Aristotle's syllogistic is suitable for the purpose. On the other hand, modern logic offers a framework for reconstructing ancient theories in their 'proper' form, as J. Łukasiewicz and his followers showed. This reconstructionist programme neatly fits the internalist taste for historiography: no matter how the Greeks 'guessed' true statements about the realm of mathematical objects, their logical gaps and errors can be eliminated by depriving their theories of the original and clumsy language of expression and, then, by reconstructing them on the correct foundations provided by modern logic.

If this is the 'right' way to do history of mathematics, then Szabó clearly misunderstood the role logic plays in this profession. Mathematicians, as Kneale warns us, do not need philosophers to tell them in advance how they should proceed in their research – logic comes into the picture later, at the stage of reconstruction. Berka arrives at the same conclusion, adding a brave remark: "There are hardly any examples in the history of mathematics that philosophers who were not at the same time mathematicians have positively influenced the development of mathematics."[8] The autonomy of mathematics must be preserved. For defenders of this view, Szabó's 'confusion' is another example of the failure of any historical explanation which calls for factors external to mathematics.

IV. Some Points of Defence

1. If, in accordance with the platonistic presupposition, the source of mathematical truth is some kind of familiarity with the independent realm of objects or facts, plus the correct deductive structure of theories is constructed only well after our knowledge of the field has been deepened enough, then we have no reason to hold that the deductive style of mathematics was 'discovered' in the same way as truths about the platonic realm were. In philosophy of science, people have long been warned again and again (if not altogether

legitimately) to distinguish the context of discovery from the context of justification. In other words, granted that the elements of our 'factual' knowledge in mathematics 'mirror' some objective realm of truth, we cannot simply also grant that proofs, and the very idea of using proofs, are rendered in the same way. The platonistic-reconstructionist set-up does not provide us with an acceptable explanation for the origin of the deductive style and its sources of development. That is, it does not even deal with the question of the origin of what it consideres ‘mathematics proper’, since it takes mathematics, in a sense, given to us once and for all – this is a price for its objection to any externalist explanatory strategy. In sum, it begs the question that Szabó raises.

2. To see how Szabó's ideas about the relation of Eleatic logic and early Greek mathematics were misunderstood, one has to appeal to a view of human knowledge different from, and richer than, the picture suggested by the formalist tradition. According to the latter, sensible knowledge must be expressed in, or even identified with, well-structured theoretical systems. Now, it is clear that Eleatic philosophers did not have a logical knowledge of this kind, and hence they were not able to build a logical metatheory of mathematics. In his criticism, Berka reminds the reader of the Greek distinction between *theoria* and *praxis*, recalling that Eleatic dialectics never reached the level of theoretical knowledge. Rather, it was a practical 'art' with proper methods and aims, but without firm theoretical foundations. What Berka does not see is that it was exactly in this capacity that Eleatic dialectics was utilised by 'mathematicians'. Eleatics developed a certain logical 'consciousness', rather than a logical theory, and they introduced some patterns of purely intellectual and anti-empirical reasoning, such as the pattern of indirect proofs. Theoretical foundations are obviously not needed for these patterns to spread over mathematical thought, should they come from dialectics or, with the same result, emerge from the genuine mathematical research. The role philosophy played in the story was surely not an intervention in the home affairs of mathematics but, as Plato's Socrates could put it, she acted as the midwife at the birth of ‘mathematics proper’.

3. The internalist school treats philosophy as an independent, separate discipline which is in any way 'external' to mathematics. This attitude is a characteristic feature of Whig-historiography whose followers take modern disciplinary boundaries for granted and project them back on the past ages of scientific inquiry. Kneale and Berka (as well as other internalists) speak of 'mathematicians' and 'philosophers', as if these categories could unproblematically be applied to the ancient Greek intellectual culture. This seems to me an especially rude instance of anachronism. There is no need to show here that in ancient Greece, everyone pursuing an intellectual understanding of the world was called 'philosophos', or 'lover of wisdom'. From Thales and the Pythagoreans to Plato's Academy, 'mathematicians' were very often 'philosophers' at the same time. During the 4th century BC,

mathematics did become an independent discipline in a sense, by gradually accepting the conceptual, methodological and terminological 'foundations' on which further systematic research could be built. But before that, a long quest for these foundations had to take place, as Szabó suggests, and not independently of philosophy in general.

With regards to the actual relation between philosophy and any scientific, understood in a broad sense, enterprise like mathematics, the philosophical approach is dominated by the sceptical attitude to re-examine and criticise every basic theoretical and conceptual commitment of our intellectual culture, while the scientific enterprise on the other hand needs some commonly accepted foundations on which it can proceed further. One of the first and most important steps in the history of philosophy was the Eleatic criticism of the sensory-empirical knowledge, and this gave birth to the purely intellectual tradition that mathematics was so successful in exploiting. However, this was not the only fruitful marriage between the two 'disciplines': all through the history of mathematics, people who introduced new conceptual fields or basic methods (like Newton, Leibniz, Cantor, Hilbert etc.) benefited from genuine philosophical investigations. Philosophy is not a separate field of knowledge from where 'external' knowledge could be imported to mathematics. Rather, it is a tool to examine, and help develop if needed, epistemic domains that we use as 'foundations', i.e. starting points, for our knowledge.

4. Finally, the internalist picture of mathematics is an ideal we have never actually met in history. If we stick to this Utopian vision of neatly formal axiomatic-deductive systems of knowledge, then we can hardly save most of the 'phenomena' in the history of mathematics. Apart from a bit of classical Greek mathematics and some of the twentieth century, mathematics was never pursued in the likeness of this ideal: rather, it seems to have been more practical and empirical. Thus, the purificationist attitude belonging to the internalist ideal of 'pure' mathematics may sacrifice too much, even in its reconstructionist (and therefore teleological) form.

Coming back to the relation between philosophy and mathematics, I suggest that it was not only the notion of philosophy which needs to be reconsidered, but also that of mathematics. I must admit that I find some of Szabó's arguments pretty convincing and, at least, I accept that in the period extending from the introduction of proof-like considerations in Greek mathematics to the acceptance of the basic elements of the axiomatic style shown in Euclid, a fruitful role was played by the Eleatic dialectic tradition and its criticism. On the other hand, one cannot deny the presence of some important and partly influential mathematical activities before the Greeks, for example, in Egypt or Babylon. My conclusion is that it is not mathematics or 'mathematics proper' that has its origins in the Eleatic anti-empirical tradition but, rather, the very ideal of 'pure' mathematics (see Plato's abstract realism and Aristotle's ideal of apodictic-deductive science); and this ideal was revived at the end of the 19th century (see all the purificationist-fundationalist schools in modern philosophy of mathematics).

Summing up, Szabó was one of the many scholars rebelling against the dominant internalist 'myth' of Pure Mathematics, together with G. Pólya and I. Lakatos, just to name two of his 'Hungarian' colleagues. Their goals have not yet been achieved but, since their time, a plurality of 'externalist' aspects has slowly been gaining authority in the general understanding of mathematics. I can only hope that this tendency will continue for a long while.

References:

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[2] S. Unguru: "On the Need to Rewrite the History of Greek Mathematics" *Archive for the History of Exact Sciences* **15** (1975), pp. 67-114. See also the interesting debate following Unguru's paper: B.L van der Waerden: "Defence of a 'Shocking' Point of View" *Archive…* **15** (1976) pp. 199-210. Or: A. Weil: "Who Betrayed Euclid?" *Archive…* **19** (1978) pp. 91-93. Unguru: "History of Ancient Mathematics: Some Reflections on the State of Art" *Isis* **70** (1979) pp. 555-565.

[3] The most fundamental work on his views is: Á. Szabó: *Anfänge der griechischen Mathematik.* Budapest, Akadémiai Kiadó – Müncen-Wien, Oldenbourg Verlag. 1969. The English version: *The Beginnings of Greek Mathematics.* Budapest, Akadémiai Kiadó – Dordrecht, Reidel Publishing Company. 1978.

[4] My translation from the following paper published in Hungarian: "A matematika terminusainak euklidészi alapjai" [Eucledian Foundations for Mathematical Terminology] In: *A görög matematika*. Magyar Tudománytörténeti Intézet, 1997, p. 134.

[5] W.C. Kneale: "Commentary to Árpád Szabó's 'Greek Dialectics and Euclid's Axiomatics'". In: I. Lakatos (ed.): *Problems in the Philosophy of Mathematics.* Amsterdam, North Holland. 1967, p. 9.

[6] R. Hersh: "Some Proposals for Reviving the Philosophy of Mathematics" *Advances in Mathematics* **31** (1979), pp. 31-50.

[7] K. Berka: "Was There an Eleatic Background to Pre-Eucledian Mathematics?" In: J. Hintikka, D. Gruender and E. Agazzi (eds.):*Theory Change, Ancient Axiomatics and Galileo's Methodology. Proceedings of the 1978 Pisa Conference on the History and Philosophy of Science.* Dordrecht-Boston-London, D. Reidel Publishing Company. 1981. Vol. 1, pp. 126-127.

[8] *Loc.cit.* p. 127.